



6.3 Linear combinations of vectors

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Introduction

- In this topic, we will
 - Review finding linear combinations of vectors equaling a target vector
 - Review solving a system of linear equations
 - Demonstrate these are equivalent problems
 - Give some examples





Abstraction

- Up to this point, we have discussed:
 - Describing vector operations
 - Specifically, describing linear combinations of vectors
 - Solving a system of linear equations
- One purpose of mathematics is to find when different problems are *essentially* the same
 - Thus, tools used to solve one problem can be used to solve the other, and vice versa
 - This is the core idea behind mathematical abstraction
 - That is, don't “reinvent the wheel”





Two problems

- Recall these two problems:
 - Find a linear combination of n vectors in $\mathbf{R}^m$ that equals a target vector $\mathbf{v}$, also in $\mathbf{R}^m$

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{v}$$

- Solve this system of m linear equations in n unknowns

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n = b_2$$

$$a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + \cdots + a_{3,n}x_n = b_3$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + a_{m,3}x_3 + \cdots + a_{m,n}x_n = b_m$$





Linear combinations to linear equations

- Starting with the first problem:
 - Find a linear combination of n vectors in $\mathbf{R}^m$ that equals a target vector $\mathbf{v}$, also in $\mathbf{R}^m$

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 + \cdots + \alpha_n \mathbf{u}_n = \mathbf{v}$$

- This is the same as asking:

$$\alpha_1 \begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ \vdots \\ u_{1,m} \end{pmatrix} + \alpha_2 \begin{pmatrix} u_{2,1} \\ u_{2,2} \\ u_{2,3} \\ \vdots \\ u_{2,m} \end{pmatrix} + \alpha_3 \begin{pmatrix} u_{3,1} \\ u_{3,2} \\ u_{3,3} \\ \vdots \\ u_{3,m} \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} u_{n,1} \\ u_{n,2} \\ u_{n,3} \\ \vdots \\ u_{n,m} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{pmatrix}$$





Linear combinations to linear equations

- Multiplying each vector by the corresponding scalar, we get

$$\begin{pmatrix} \alpha_1 u_{1,1} \\ \alpha_1 u_{1,2} \\ \alpha_1 u_{1,3} \\ \vdots \\ \alpha_1 u_{1,m} \end{pmatrix} + \begin{pmatrix} \alpha_2 u_{2,1} \\ \alpha_2 u_{2,2} \\ \alpha_2 u_{2,3} \\ \vdots \\ \alpha_2 u_{2,m} \end{pmatrix} + \begin{pmatrix} \alpha_3 u_{3,1} \\ \alpha_3 u_{3,2} \\ \alpha_3 u_{3,3} \\ \vdots \\ \alpha_3 u_{3,m} \end{pmatrix} + \cdots + \begin{pmatrix} \alpha_n u_{n,1} \\ \alpha_n u_{n,2} \\ \alpha_n u_{n,3} \\ \vdots \\ \alpha_n u_{n,m} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{pmatrix}$$





Linear combinations to linear equations

- Adding the vectors yields

$$\begin{pmatrix} \alpha_1 u_{1,1} + \alpha_2 u_{2,1} + \alpha_3 u_{3,1} + \cdots + \alpha_n u_{n,1} \\ \alpha_1 u_{1,2} + \alpha_2 u_{2,2} + \alpha_3 u_{3,2} + \cdots + \alpha_n u_{n,2} \\ \alpha_1 u_{1,3} + \alpha_2 u_{2,3} + \alpha_3 u_{3,3} + \cdots + \alpha_n u_{n,3} \\ \vdots \\ \alpha_1 u_{1,m} + \alpha_2 u_{2,m} + \alpha_3 u_{3,m} + \cdots + \alpha_n u_{n,m} \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{pmatrix}$$





Linear combinations to linear equations

- Two vectors are equal if and only if their corresponding entries are all equal, so

$$\alpha_1 u_{1,1} + \alpha_2 u_{2,1} + \alpha_3 u_{3,1} + \cdots + \alpha_n u_{n,1} = v_1$$

$$\alpha_1 u_{1,2} + \alpha_2 u_{2,2} + \alpha_3 u_{3,2} + \cdots + \alpha_n u_{n,2} = v_2$$

$$\alpha_1 u_{1,3} + \alpha_2 u_{2,3} + \alpha_3 u_{3,3} + \cdots + \alpha_n u_{n,3} = v_3$$

$$\vdots$$

$$\alpha_1 u_{1,m} + \alpha_2 u_{2,m} + \alpha_3 u_{3,m} + \cdots + \alpha_n u_{n,m} = v_m$$

$$v_k = b_k$$

$$\alpha_\ell = x_\ell$$

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n = b_2$$

$$a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + \cdots + a_{3,n}x_n = b_3$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + a_{m,3}x_3 + \cdots + a_{m,n}x_n = b_m$$

$$u_{\ell,k} = a_{k,\ell}$$

$$\alpha_\ell u_{\ell,k} = a_{k,\ell} x_\ell$$





Linear equations to linear combinations

- Starting with a system of m linear equations in n unknowns

$$a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n = b_2$$

$$a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + \cdots + a_{3,n}x_n = b_3$$

$$\vdots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + a_{m,3}x_3 + \cdots + a_{m,n}x_n = b_m$$





Linear equations to linear combinations

- Let us make each of these the entries of two vectors in $\mathbf{R}^m$

$$\begin{pmatrix} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 + \cdots + a_{2,n}x_n \\ a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 + \cdots + a_{3,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + a_{m,3}x_3 + \cdots + a_{m,n}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix}$$





Linear equations to linear combinations

- Separating this into a sum of n vectors, we have

$$\begin{pmatrix} a_{1,1}x_1 \\ a_{2,1}x_1 \\ a_{3,1}x_1 \\ \vdots \\ a_{m,1}x_1 \end{pmatrix} + \begin{pmatrix} a_{1,2}x_2 \\ a_{2,2}x_2 \\ a_{3,2}x_2 \\ \vdots \\ a_{m,2}x_2 \end{pmatrix} + \begin{pmatrix} a_{1,3}x_3 \\ a_{2,3}x_3 \\ a_{3,3}x_3 \\ \vdots \\ a_{m,3}x_3 \end{pmatrix} + \cdots + \begin{pmatrix} a_{1,n}x_n \\ a_{2,n}x_n \\ a_{3,n}x_n \\ \vdots \\ a_{m,n}x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix}$$





Linear equations to linear combinations

- Removing the common unknown scalar x_k , we have

$$x_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ a_{3,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + x_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \\ a_{3,2} \\ \vdots \\ a_{m,2} \end{pmatrix} + x_3 \begin{pmatrix} a_{1,3} \\ a_{2,3} \\ a_{3,3} \\ \vdots \\ a_{m,3} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ a_{3,n} \\ \vdots \\ a_{m,n} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix}$$





Linear equations to linear combinations

- If we define

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix} \text{ and } \mathbf{a}_\ell = \begin{pmatrix} a_{1,\ell} \\ a_{2,\ell} \\ a_{3,\ell} \\ \vdots \\ a_{m,\ell} \end{pmatrix}$$

- Thus, we have

$$x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + x_3 \mathbf{a}_3 + \cdots + x_n \mathbf{a}_n = \mathbf{b}$$





Equivalence

- Thus, every question of finding a linear combination of a collection of n vectors equaling a target vector all in $\mathbf{R}^m$ defines a question of solving a system of m linear equations in n unknowns
- Similarly, every question of solving a system of m linear equations in n unknowns defines a question of finding a linear combination of n vectors equaling a target vector all in $\mathbf{R}^m$
 - Thus, these problems are equivalent, so any mechanism for solving the first also solves the second, and vice versa





Example

- Convert the following system of two linear equations in three unknowns into a linear combination of vectors equaling a target vector

$$4x_1 + 2x_2 - 5x_3 = 7$$

$$-8x_1 - 6x_3 = 9$$

$$x_1 \begin{pmatrix} 4 \\ -8 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ -6 \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} -1.125 - 0.75x_3 \\ 5.75 + 4x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1.125 \\ 5.75 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -0.75 \\ 4 \\ 1 \end{pmatrix}$$





Example

- Convert the following system of four linear equations in two unknowns into a linear combination of vectors equaling a target vector

$$3x_1 + x_2 = 5$$

$$-x_1 = 9$$

$$2x_1 + 7x_2 = -4$$

$$-6x_1 + 8x_2 = 0$$

$$x_1 \begin{pmatrix} 3 \\ -1 \\ 2 \\ -6 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \\ 7 \\ 8 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ -4 \\ 0 \end{pmatrix}$$





Example

- Convert the following system of three linear equations in three unknowns into a linear combination of vectors equaling a target vector

$$x_1 + 4x_2 + 7x_3 = 0$$

$$2x_1 + 5x_2 + 8x_3 = 0$$

$$3x_1 + 6x_2 + 9x_3 = 0$$

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + x_3 \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$





Example

- Convert the following question of finding a linear combination of two vectors in $\mathbf{R}^3$ equaling a target vector into a system of linear equations

$$\alpha_1 \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} + \alpha_2 \begin{pmatrix} -3 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ -13 \\ 7 \end{pmatrix}$$

$$\alpha_1 - 3\alpha_2 = 8$$

$$-2\alpha_1 + 4\alpha_2 = -13$$

$$5\alpha_1 + 7\alpha_2 = 7$$

$$\boldsymbol{\alpha} = \begin{pmatrix} 3.5 \\ -1.5 \end{pmatrix}$$





Example

- Convert the following question of finding a linear combination of two vectors in $\mathbf{R}^2$ equaling a target vector into a system of linear equations

$$\alpha_1 \begin{pmatrix} 3 \\ -6 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$$

$$3\alpha_1 - \alpha_2 = 5$$

$$-6\alpha_1 + 2\alpha_2 = -12$$





Summary

- Following this topic, you now
 - Understand that the two presented problems are equivalent
 - Know we will find tools for solving the first,
and this will give us tools for solving the second





References

- [1] https://en.wikipedia.org/wiki/System_of_linear_equations





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Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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